Necessary conditions for classical super-integrability of a certain family of potentials in constant curvature spaces

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Abstract. We formulate the necessary conditions for the maximal super-integrability of a certain family of classical potentials defined in the constant curvature two-dimensional spaces. We give examples of homogeneous potentials of degree -2 on $\mathbb{E}^2$ as well as their equivalents on $\mathbb{S}^2$ and $\mathbb{H}^2$ for which these necessary conditions are also sufficient. We show explicit forms of the additional first integrals which always can be chosen polynomial with respect to the momenta and which can be of an arbitrary high degree with respect to the momenta.

Key words: super-integrable systems, integrability, Hamiltonian equations, differential Galois integrability obstructions.

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1 Introduction

In this paper we consider classical Hamiltonian systems with n degrees of freedom given by Hamiltonian function $H(\boldsymbol{q}, \boldsymbol{p})$, where $\boldsymbol{q} = (q_1, \ldots, q_n)$ are canonical coordinates and $\boldsymbol{p} = (p_1, \ldots, p_n)$ are the canonical momenta. We say that such a system is maximally super-integrable if the Hamilton's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i = \frac{\partial H}{\partial p_i}(\boldsymbol{q}, \boldsymbol{p}), \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_i = -\frac{\partial H}{\partial q_i}(\boldsymbol{q}, \boldsymbol{p}), \qquad i = 1, \ldots, n, \tag{1.1}$$

admit $2n-1$ functionally independent first integrals such that among them n commute. Although such systems are highly exceptional they attract a lot of attentions. Motivation for study of super-integrable systems comes from the classical as well as from quantum physics, see, e.g., Tempesta et al. [2004]. Classical maximal integrability implies that all bounded trajectories are closed and the motion is periodic. In quantum mechanics maximal super-integrability means the existence of $2n-1$ well defined, algebraically independent operators (including Hamiltonian) among them n pairwise commute see e.g. Gravel and Winternitz [2002], Rodriguez and Winternitz [2002] and references therein. For quantum systems the maximal super-integrability implies the degeneracy of energy levels. The problem of construction of integrable, super-integrable and maximally super-integrable quantum systems from the corresponding classical ones is very complicated, see e.g. Hietarinta [1984], Gravel and Winternitz [2002], and it will be not considered in this paper.

Recently, some remarkable families of super-integrable systems were found. In Tremblay et al. [2009], Tremblay et al. [2010] the authors introduced a family of quantum and classical systems for which the classical Hamiltonian function in polar coordinates (r,φ) is given by

$$H_n^{(0)} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V_n^{(0)}(r,\varphi), \tag{1.2}$$

where the potential $V_n^{(0)}(r,\varphi)$ has the form

$$V_n^{(0)}(r,\varphi) := \frac{a}{r^2\cos^2(n\varphi)} + \frac{b}{r^2\sin^2(n\varphi)}, \tag{1.3}$$

n is an integer, a, and b are parameters[1]. The system is integrable and it has the following first integral

$$G := \frac{1}{2}p_\varphi^2 + r^2 V_n^{(0)}(r,\varphi). \tag{1.4}$$

In quantum version momenta are replaced by appropriate partial derivative operators. As it was shown in Tremblay et al. [2009], for small integer values of n the quantum system is super-integrable, and moreover the degree with respect to the momenta of the second additional first integral grows with n. On this basis a conjecture that the system is super-integrable for an arbitrary n was formulated. Later it was justified that in fact the quantum system is super-integrable for any integer odd n in Quesne [2010] and even for rational n in Kalnins et al. [2010b]. For the classical system in Tremblay et al. [2010] it was shown that all bounded trajectories are closed for all integer and rational values of n and in Kalnins et al. [2010a] forms of first integrals were described.

The second example needs a more detailed presentation. We consider a point with the unit mass moving on a sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ under influence of Hooke forces. A Hooke centre located at point $\boldsymbol{r} \in \mathbb{S}^2$ generates a potential field of forces. The value of the potential at point $\boldsymbol{\gamma} \in \mathbb{S}^2$ is

$$V = \frac{\alpha}{(\boldsymbol{\gamma}\cdot\boldsymbol{r})^2}, \tag{1.5}$$

[1] In the cited paper the potential function contains the harmonic oscillator term $\omega^2 r^2/2$.

where α is the intensity of the centre. In Borisov et al. [2009] the authors investigated the problem of a point mass moving on a sphere $\mathbb{S}^2$ in the field of odd number $n = 2l + 1$ of Hooke centres with equal intensities located in a great circle at the vortexes of the regular n-gon. We can assume that the Hooke centres are located at points

$$\boldsymbol{r}_{k;n} = (\sin\varphi_{k;n}, \cos\varphi_{k;n}, 0), \quad \text{where} \quad \varphi_{k;n} := \frac{2\pi k}{n} \quad \text{for} \quad k = 1, \dots, n. \tag{1.6}$$

Thus, the potential has the form

$$V_n^{(1)} := \sum_{k=1}^{n} \frac{\alpha}{\sin^2\theta \sin^2(\varphi + \varphi_{k;n})} = \frac{\alpha n^2}{\sin^2\theta \sin^2 n\varphi}. \tag{1.7}$$

In the above we used the following trigonometric identity

$$\sum_{k=1}^{n} \frac{1}{\sin^2(\varphi + \varphi_{k;n})} = \frac{n^2}{\sin^2 n\varphi},$$

see e.g. Jakubský et al. [2005].

Thus denoting $a := \alpha n^2$, we can write the Hamiltonian of the system in the following form

$$H_n^{(1)} = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) + \frac{a}{\sin^2\theta \sin^2 n\varphi}. \tag{1.8}$$

The system is integrable and the following function

$$F_1 := \frac{1}{2}p_\varphi^2 + \frac{a}{\sin^2 n\varphi} \tag{1.9}$$

is a first integral. In Borisov et al. [2009] it was shown that this Hamiltonian system is superintegrable as it possesses the second additional first integral of degree either $2n+1$ or $2n+2$ with respect to the momenta. The proof of this fact is remarkably natural and simply. We show later that it allows to give several generalisations of the above two examples.

The above examples have the same, in some sense nature. In Cartesian coordinates potential (1.3) has the form

$$V_n^{(0)} = \frac{a(q_1^2+q_2^2)^{n-1}}{[\mathrm{Re}(q_1 + \mathrm{i}q_2)^n]^2} + \frac{b(q_1^2+q_2^2)^{n-1}}{[\mathrm{Im}(q_1 + \mathrm{i}q_2)^n]^2}. \tag{1.10}$$

Hence it is a rational homogeneous function of degree -2. As it is well known a natural Hamiltonian system given by

$$H^{(0)} := \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{1.11}$$

with a homogeneous potential of degree -2 is integrable because it possesses the following first integral

$$F_1 := \frac{1}{2}(q_1 p_2 - q_2 p_1)^2 + (q_1^2 + q_2^2)V(q_1, q_2). \tag{1.12}$$

For a point on a sphere we have analogous potential. Namely, Hamiltonian system given by

$$H_n^{(1)} = \frac{1}{2}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) + \frac{1}{\sin^2\theta}U(\varphi), \tag{1.13}$$

is integrable with the first integral

$$F_1 := \frac{1}{2}p_\varphi^2 + U(\varphi). \tag{1.14}$$

Considering the above two examples, we can ask whether a potential of the prescribed form is super-integrable. It appears that this question is difficult if we look for an effective and computable necessary conditions for the super-integrability.

In this paper we consider natural Hamiltonian systems with two degrees of freedom defined on $T^\star M$ where M is a two dimensional manifold with a constant curvature metrics. More specifically, M is either sphere $\mathbb{S}^2$, Euclidean plane $\mathbb{E}^2$, or the hyperbolic plane $\mathbb{H}^2$. In order to consider those three cases simultaneously we will proceed as in Herranz et al. [2000], Rañada and Santander [1999] and we define the following functions

$$\mathrm{C}_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ 1 & \text{for} \quad \kappa = 0, \\ \cosh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0, \end{cases} \tag{1.15}$$

$$\mathrm{S}_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ x & \text{for} \quad \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0. \end{cases} \tag{1.16}$$

These functions satisfy the following identities

$$\mathrm{C}_\kappa^2(x) + \kappa\,\mathrm{S}_\kappa^2(x) = 1, \quad \mathrm{S}_\kappa'(x) = \mathrm{C}_\kappa(x), \quad \mathrm{C}_\kappa'(x) = -\kappa\,\mathrm{S}_\kappa(x). \tag{1.17}$$

We consider natural systems Hamiltonian systems with potential $V(r,\varphi)$ defined by

$$H^{(\kappa)} = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathrm{S}_\kappa^2(r)}\right) + V(r,\varphi). \tag{1.18}$$

The form of the kinetic energy corresponds to the metric on M with constant curvature κ. Our aim is to distinguish a special class of super-integrable potentials. Inspired by the examples discussed above we consider potentials of the form

$$V^{(\kappa)}(r,\varphi) := \frac{1}{\mathrm{S}_\kappa^2(r)}U(\varphi). \tag{1.19}$$

These potentials are separable. In fact

$$G := \frac{1}{2}p_\varphi^2 + U(\varphi), \tag{1.20}$$

is a first integral of the system and we have also

$$H = \frac{1}{2}p_r^2 + \frac{1}{\mathrm{S}_\kappa^2(r)}G. \tag{1.21}$$

In order to formulate our main result let us assume that there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. Under this assumption we define the following quantity

$$\lambda := 1 - \frac{1}{2}\frac{U''(\varphi_0)}{U(\varphi_0)}. \tag{1.22}$$

The most important result of this paper is formulated in the following theorem which gives necessary conditions for the super-integrability of systems (1.21) with potential (1.19).

Theorem 1.1. *Assume that potential $V^{(\kappa)}$ given by* (1.19) *satisfies the following assumption: there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. If $V^{(\kappa)}$ is super-integrable, then*

$$\lambda := 1 - \frac{1}{2}\frac{U''(\varphi_0)}{U(\varphi_0)} = 1 - s^2,$$

for a certain non-zero rational number s.

The necessary conditions for the super-integrability given by Theorem 1.1 are deduced from an analysis of the differential Galois group of the variational equations along the described particular solution. Here we refer to our paper Maciejewski et al. [2008] where the reader will find a description of applications of the differential Galois theory to a study of the integrability and the super-integrability as well as an analysis of the case $\kappa = 0$. Indeed the statement of Theorem 1.1 for the case $\kappa = 0$ is just a rephrase of the previous result written in polar coordinates, as it will be seen in the next section.

2 Relation with known necessary conditions for super-integrability

For a systems on $\mathbb{E}^2$ given by natural Hamiltonian (1.11) with a homogeneous potential $V(q_1, q_2)$ of degree k equations of motion have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i = p_i, \qquad \frac{\mathrm{d}}{\mathrm{d}t}p_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2. \tag{2.1}$$

One can look for their particular solution of the form $\boldsymbol{q}(t) = \varphi(t)\boldsymbol{c}$, $\boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{c}$, where $\boldsymbol{c} \in \mathbb{C}^2$ is a non-zero vector, and $\varphi(t)$ is a scalar function. As it is easy to see such a solution exists provided that $\boldsymbol{c}$ is a non-zero solution of $\operatorname{grad} V(\boldsymbol{c}) = \boldsymbol{c}$, and $\varphi(t)$ satisfies $\ddot{\varphi} + \varphi^{k-1} = 0$. Vector $\boldsymbol{c}$ is called the Darboux point of potential V. Then the necessary conditions for the integrability and the super-integrability which come from an analysis of the differential Galois group of the variational equations along the described particular solution are expressed by means of one eigenvalue of the Hessian matrix $V''(\boldsymbol{c})$, see Morales [1999] for the integrability obstructions, and Maciejewski et al. [2008] for super-integrability conditions. For matrix $V''(\boldsymbol{c})$ vector $\boldsymbol{c}$ is an eigenvector with the corresponding eigenvalue $(k-1)$. Thus the other eigenvalue is given by

$$\lambda = \operatorname{Tr} V''(\boldsymbol{c}) - (k-1) = \nabla^2 V(\boldsymbol{c}) - (k-1). \tag{2.2}$$

The mentioned above necessary conditions for the integrability have the form of arithmetic restrictions imposed on λ. For example, in our previous paper Maciejewski et al. [2008], we proved, among other things the following.

Theorem 2.1. *If a Hamiltonian system given by* (1.11)*, with a homogeneous potential $V(q_1, q_2)$ of degree k, $|k| \leq 2$ is super-integrable, then for each Darboux point $\boldsymbol{c}$ the corresponding eigenvalue λ satisfies the following conditions:*

- *if $k = 2$, then $\lambda = s^2$, where s is a non-zero rational number;*
- *if $k = 1$, then $\lambda = 0$;*
- *if $k = -1$, then $\lambda = 1$;*
- *if $k = -2$, then $\lambda = 1 - s^2$, where s is a non-zero rational number.*

In the polar coordinates homogeneous potential have the form

$$V(\boldsymbol{q}) = V(r\cos\varphi, r\sin\varphi) = r^k U(\varphi), \tag{2.3}$$

and a Darboux point is given by

$$(c_1, c_2) = c(\cos\varphi_0, \sin\varphi_0), \tag{2.4}$$

where φ_0 is a solution of $U'(\varphi) = 0$ such that $U(\varphi_0) \neq 0$.

The Laplacian $\nabla^2 V$ of function $V(\boldsymbol{q})$ takes the form

$$\nabla^2 V = \frac{\partial^2 V}{\partial q_1^2} + \frac{\partial^2 V}{\partial q_2^2} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \varphi^2}$$

in polar coordinates, and for $V = r^k U(\varphi)$,

$$\nabla^2 V = k^2 r^{k-2} U(\varphi) + r^{k-2} U''(\varphi).$$

Thus, as computed by Sansaturio et al. [1997], we have

$$\lambda = k^2 c^{k-2} U(\varphi_0) + c^{k-2} U''(\varphi_0) - (k-1) = 1 + c^{k-2} U''(\varphi_0) = 1 + \frac{U''(\varphi_0)}{k\, U(\varphi_0)}, \tag{2.5}$$

and substitution $k = -2$ reproduces λ in (1.22). Therefore the statement of Theorem 2.1 with $k = -2$ gives Theorem 1.1 immediately. So the novelty of Theorem 1.1 is to confirm that the same statement holds, independent of the value of the curvature κ.

3 Proof of Theorem 1.1

According to Theorem 1.2 in Maciejewski et al. [2008], if the considered system is maximally super-integrable, then the identity component the differential Galois group of the normal variational equations along a particular solution is just the identity.

The assumptions in Theorem 1.1 guarantee that

$$t \longmapsto (\varphi_0, r(t), 0, \dot{r}(t)) \tag{3.1}$$

is a particular solution of the system provided that $r(t)$ satisfies

$$\ddot{r} = 2U(\varphi_0)\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)^3}. \tag{3.2}$$

We consider a particular solution with the energy e, i.e., we fix

$$e = \frac{1}{2}\dot{r}^2 + \frac{U(\varphi_0)}{\mathrm{S}_\kappa(r)^2}. \tag{3.3}$$

Variational equations along particular solution (3.1) have the form

$$\begin{bmatrix} \dot{R} \\ \dot{\Phi} \\ \dot{P}_R \\ \dot{P}_\Phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{S}_\kappa^{-2}(r) \\ 2\,\mathrm{S}_\kappa^{-4}(r)[2\kappa\,\mathrm{S}_\kappa^2(r) - 3]U(\varphi_0) & 0 & 0 & 0 \\ 0 & -\,\mathrm{S}_\kappa^{-2}(r)U''(\varphi_0) & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ \Phi \\ P_R \\ P_\Phi \end{bmatrix}. \tag{3.4}$$

Since the motion takes place in the plane (r, p_r) the normal part of variational equations is

$$\dot{\Phi} = \mathrm{S}_\kappa^{-2}(r)P_\Phi, \qquad \dot{P}_\Phi = -\,\mathrm{S}_\kappa^{-2}(r)U''(\varphi_0)\Phi.$$

We rewrite this system as one equation of the second order

$$\ddot{\Phi} + a(r, p_r)\dot{\Phi} + b(r, p_r)\Phi = 0,$$

where

$$a(r, p_r) = 2\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}p_r, \quad b(r, p_r) = \mathrm{S}_\kappa^{-4}(r)U''(\varphi_0).$$

Making the following change of the independent variable $t \mapsto s = \mathrm{S}_\kappa(r)$, we transform this equation into a linear equation with rational coefficients

$$\Phi'' + p(s)\Phi' + q(s)\Phi = 0, \tag{3.5}$$

where

$$\begin{aligned} p(s) &= \frac{\ddot{s} + \dot{s}a}{\dot{s}^2} = \frac{1}{s} + \frac{es}{es^2 - B} + \frac{\kappa s}{\kappa s^2 - 1}, \\ q(s) &= \frac{b}{\dot{s}^2} = \frac{B(\lambda - 1)}{s^2(es^2 - B)(\kappa s^2 - 1)}, \qquad B := U(\varphi_0). \end{aligned} \tag{3.6}$$

Finally, we perform one more transformation of the independent variable putting

$$z = \frac{es^2 - B}{(e - \kappa B)s^2}. \tag{3.7}$$

After this transformation equation (3.5) reads

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + P\frac{\mathrm{d}\Phi}{\mathrm{d}z} + Q\Phi = 0, \tag{3.8}$$

where

$$P = \frac{2z-1}{2z(z-1)}, \qquad Q := \frac{\lambda-1}{4z(z-1)}. \tag{3.9}$$

This is the Gauss hypergeometric equation for which the differences of exponents at $z=0$, $z=1$ and $z=\infty$ are

$$\rho = \frac{1}{2}, \qquad \sigma = \frac{1}{2}, \qquad \tau = \sqrt{1-\lambda}, \tag{3.10}$$

respectively. Putting

$$w = \Phi \exp\left[\frac{1}{2}\int P(\zeta)\mathrm{d}\zeta\right], \tag{3.11}$$

we transform this equation to its reduced form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = r(z)w, \tag{3.12}$$

where

$$r(z) = -\frac{4\lambda z(z-1)+3}{16z^2(z-1)^2}. \tag{3.13}$$

This equation coincides with equation (A.9) in Maciejewski et al. [2008] in which we substitute $k=-2$. Thus, we can apply Proposition A.3 and Proposition A.4 from Maciejewski et al. [2008] to equation (3.12), and this exactly gives the thesis of our theorem.

4 Tremblay-Turbiner-Winternitz (TTW) system: A family of super-integrable systems in Euclidean flat space $\mathbb{E}^2$

4.1 Finding the potential by use of the necessary conditions for super-integrability

In general, values of λ in (3.12) can depend on the angle φ_0, or, on the Darboux point, and in order to be super-integrable, the only necessary condition is that $\lambda = 1-s^2$, for a certain non-zero rational number s. However, there is a special class of super-integrable potentials, for which λ takes the same admissible value for all choices of the angle φ_0. Indeed, the known super-integrable potential (1.3) is strongly characterised by possessing this property as it is shown below.

First, condition $\lambda = 1-s^2$ with expression (1.22) gives

$$-\frac{1}{2}\frac{U''(\varphi_0)}{U(\varphi_0)} = -s^2. \tag{4.1}$$

Let us change the dependent variable $U(\varphi)$, or the angular part of the potential, by

$$U(\varphi) = \frac{1}{[f(\varphi)]^2}.$$

Then at a point where $U'(\varphi_0)=0$, we have $f'(\varphi_0)=0$, and furthermore,

$$-\frac{1}{2}\frac{U''(\varphi_0)}{U(\varphi_0)} = \frac{f''(\varphi_0)}{f(\varphi_0)}. \tag{4.2}$$

Next we force the additional requirement that λ takes the same value $\lambda = 1 - s^2$, for all Darboux points $U'(\varphi_0) = 0$. That is, we have the relation

$$f''(\varphi_0) = -s^2 f(\varphi_0), \tag{4.3}$$

for all φ_0 such that $f'(\varphi_0)$ vanishes. Here s is a non-zero rational number, which is independent of φ_0. This requirement does not determine function f uniquely. However, we can find examples of f satisfying this condition. The simplest and naive example is the function f which satisfies the differential equation

$$f''(\varphi) = -s^2 f(\varphi). \tag{4.4}$$

This is equivalent to assume that f satisfies (4.3) for all $\varphi_0 \in (0, 2\pi)$. Then we find two independent solutions for f

$$f_1(\varphi) = \cos(s\varphi), \quad f_2(\varphi) = \sin(s\varphi)$$

and therefore

$$U_1(\varphi) = \frac{1}{\cos^2(s\varphi)}, \quad U_2(\varphi) = \frac{1}{\sin^2(s\varphi)}. \tag{4.5}$$

are the desired angular part of the super-integrable potential. Note that any linear combination of $U_1(\varphi)$ and $U_2(\varphi)$, namely

$$U = aU_1(\varphi) + bU_2(\varphi) = \frac{a}{\cos^2(s\varphi)} + \frac{b}{\sin^2(s\varphi)} \tag{4.6}$$

has the same property, but unless $ab = 0$ the value of λ is $\lambda = 1 - (2s)^2$, instead of $\lambda = 1 - s^2$. Indeed, for the potential (1.3) we have the following

Lemma 4.1. *If $\varphi_0 \in \mathbb{C}$ is a solution of equation $U'(\varphi) = 0$ such that $U(\varphi_0) \neq 0$, then*

$$\lambda = 1 - \frac{1}{2}\frac{U''(\varphi_0)}{U(\varphi_0)} = \begin{cases} 1 - n^2, & \text{if} \quad ab = 0, \\ 1 - (2n)^2, & \text{otherwise}. \end{cases} \tag{4.7}$$

Proof. For the potential (1.3) we have

$$U(\varphi) = \frac{a}{\cos^2(n\varphi)} + \frac{b}{\sin^2(n\varphi)}, \tag{4.8}$$

so

$$U'(\varphi) = 2na\frac{\sin(n\varphi)}{\cos^3(n\varphi)} - 2nb\frac{\cos(n\varphi)}{\sin^3(n\varphi)}, \tag{4.9}$$

and

$$U''(\varphi) = 2n^2 a\frac{1 + 2\sin^2(n\varphi)}{\cos^4(n\varphi)} + 2n^2 b\frac{1 + 2\cos^2(n\varphi)}{\sin^4(n\varphi)}. \tag{4.10}$$

Let us assume that $ab \neq 0$. In a case when $a \neq 0$ and $b = 0$, from (4.9) we have $\sin(n\varphi_0) = 0$. Thus, $U(\varphi_0) = a$, and, by (4.10), $U''(\varphi_0) = 2n^2 a$, so we have relation (4.7). In a similar way we show that this formula is valid in the case $a = 0$ and $b \neq 0$.

If $ab \neq 0$, then from (4.9) we find that

$$b = a \tan^4(n\varphi_0).$$

Using this relation we obtain

$$U(\varphi_0) = \frac{a}{\cos^4(n\varphi_0)}, \quad \text{and} \quad U''(\varphi_0) = \frac{8n^2 a}{\cos^4(n\varphi_0)},$$

and this finishes the proof. □

4.2 Checking the super-integrability by separation of variables

As we have shown that if n is a non-zero rational number, then potential (1.3) satisfies the necessary condition for the super-integrability, next we are going to prove that this potential is indeed super-integrable. We demonstrate this giving an explicit form of the second additional first integral F which is functionally independent together with $H_n^{(0)}$ and G given by (1.2) and (1.4), respectively. However, in order to demonstrate how to derive this first integral we consider at first simplified case when $b = 0$ in potential (1.3). Then Hamiltonian takes the form

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \frac{a}{r^2\cos^2(n\varphi)} = \frac{p_r^2}{2} + \frac{1}{r^2}G, \tag{4.11}$$

where G is the first integral given by

$$G = \frac{p_\varphi^2}{2} + \frac{a}{\cos^2(n\varphi)}. \tag{4.12}$$

In order to perform the explicit integration we introduce as in Borisov et al. [2009] a new independent variable τ such that $\mathrm{d}\tau/\mathrm{d}t = 1/r^2$. Then we find that

$$p_r = \frac{r'}{r^2}, \qquad p_\varphi = \varphi',$$

where prime denotes the differentiation with respect to τ. In effect we have

$$H = \frac{r'^2}{2r^4} + \frac{1}{r^2}G, \quad \text{and} \quad G = \frac{\varphi'^2}{2} + \frac{a}{\cos^2(n\varphi)},$$

i.e., we effectively separated variables

$$\int \frac{\mathrm{d}r}{r\sqrt{2(Hr^2 - G)}} = \tau + C_1, \qquad \int \frac{\cos(n\varphi)\mathrm{d}\varphi}{\sqrt{2(G\cos^2(n\varphi) - a)}} = \tau + C_2. \tag{4.13}$$

The explicit forms of these elementary integrals are following

$$\frac{1}{\sqrt{2G}} \arctan \sqrt{\frac{Hr^2 - G}{G}} = \tau + C_1, \tag{4.14}$$

and

$$\frac{1}{n\sqrt{2G}}\arcsin\left[\sqrt{\frac{G}{G-a}}\sin(n\varphi)\right]=\tau+C_2. \tag{4.15}$$

From (4.14) and (4.15) we deduce that

$$I=n\sqrt{2G}(C_2-C_1)=\arcsin\left[\sqrt{\frac{G}{G-a}}\sin(n\varphi)\right]-n\arctan\sqrt{\frac{Hr^2-G}{G}}, \tag{4.16}$$

is a first integral of the system. Using it we find an algebraic first integral. To this end we perform a sequence of transformations applying the following formulae

$$\arcsin z=-\mathrm{i}\ln\left(\mathrm{i}z+\sqrt{1-z^2}\right),\ \arccos z=-\mathrm{i}\ln\left(z+\sqrt{z^2-1}\right),\ \arctan z=\frac{\mathrm{i}}{2}\ln\left(\frac{1-\mathrm{i}z}{1+\mathrm{i}z}\right).$$

Using them and making some simplifications we obtain

$$I=-\mathrm{i}\ln\left\{\left(\frac{p_\varphi\cos(n\varphi)}{\sqrt{2(G-a)}}+\mathrm{i}\sqrt{\frac{G}{G-a}}\sin(n\varphi)\right)\frac{(\sqrt{2G}-\mathrm{i}rp_r)^n}{(2H)^{n/2}r^n}\right\}.$$

From the above formula we deduce that

$$\begin{aligned} I_1&=(2H)^{n/2}\sqrt{2(G-a)}\exp(\mathrm{i}\,I)\\ &=\frac{1}{r^n}\left(p_\varphi\cos(n\varphi)+\mathrm{i}\sqrt{2G}\sin(n\varphi)\right)(\sqrt{2G}-\mathrm{i}rp_r)^n, \end{aligned} \tag{4.17}$$

is a first integral of the system. For rational n this integral is an algebraic function of Cartesian variables (q_1,q_2,p_1,p_2).

If the considered system is real, then one would like to possess real first integrals. Taking the real and imaginary parts of I_1 (assuming that all variables are real) we obtain real first integrals. Let us assume for simplicity that n is a positive integer. Then

$$I_1=r^{-n}\left(p_\varphi\cos(n\varphi)+\mathrm{i}\sqrt{2G}\sin(n\varphi)\right)\sum_{k=0}^{n}\binom{n}{k}(2G)^{(n-k)/2}(-\mathrm{i})^k r^k p_r^k,$$

and from this we obtain

$$\begin{aligned} F_1=\operatorname{Re}I_1&=\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}(2G)^{\frac{n-2k}{2}}\frac{p_r^{2k}}{r^{n-2k}}\left[p_\varphi\cos(n\varphi)+\frac{n-2k}{2k+1}rp_r\sin(n\varphi)\right],\\ F_2=\operatorname{Im}I_1&=\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}(2G)^{\frac{n-2k-1}{2}}\frac{p_r^{2k}}{r^{n-2k}}\left[2G\sin(n\varphi)-\frac{n-2k}{2k+1}rp_rp_\varphi\cos(n\varphi)\right]. \end{aligned} \tag{4.18}$$

Here $[x]$ denotes the integer part of x. We note that always one of these first integrals is a polynomial in momenta (p_r,p_φ). For n even expression $(2G)^{(n-2k)/2}$ is a polynomial and as result F_1 is a polynomial in the momenta. Similarly one can deduce that for n odd F_2 is a polynomial in the momenta. Let us note that if we put negative n in (4.11), then the potential does not change, thus we can assume that always $n>0$. The same is true also for more general form of potential (1.3).

For positive rational $n = n_1/n_2$ from (4.17) also a polynomial in the momenta first integral can be constructed. Namely we consider the new first integral

$$I_2 := I_1^{n_2} = r^{-n_1}\left(p_\varphi \cos(n\varphi) + \mathrm{i}\sqrt{2G}\sin(n\varphi)\right)^{n_2}(\sqrt{2G} - \mathrm{i}rp_r)^{n_1}.$$

Separating real and imaginary parts of this first integral we find that for n_1 even and n_2 odd integral $F_1 := \operatorname{Re} I_2$ is polynomial in momenta. Moreover, for odd n_1 integral $F_2 := \operatorname{Im} I_2$ is polynomial in momenta independently of the parity of n_2.

The described direct approach works perfectly in the same way for the general form of the potential (1.3) and it gives the following form of the first integral

$$I_1 = r^{-2n}(\sqrt{2G} - \mathrm{i}rp_r)^{2n}\left[\sqrt{2G}\sin(2n\varphi)p_\varphi - 2\mathrm{i}(G\cos(2n\varphi) + b - a)\right]. \tag{4.19}$$

Assuming that n is a positive integer, then

$$I_1 = r^{-2n}\left[\sqrt{2G}\sin(2n\varphi)p_\varphi - 2\mathrm{i}(G\cos(2n\varphi) + b - a)\right]\sum_{k=0}^{2n}\binom{2n}{k}(2G)^{(2n-k)/2}(-\mathrm{i})^k r^k p_r^k, \tag{4.20}$$

and the real and imaginary parts of this complex function give additional first integrals

$$\begin{aligned}
F_1 &= \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}(2G)^{n-k}\frac{p_r^{2k}}{r^{2(n-k)}}\left[G\sin(2n\varphi)p_\varphi - \frac{2(n-k)}{2k+1}(G\cos(2n\varphi) + b - a)rp_r\right],\\
F_2 &= \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}(2G)^{n-k}\frac{p_r^{2k}}{r^{2(n-k)}}\left[\frac{2(n-k)}{2k+1}\sin(2n\varphi)rp_rp_\varphi + 2(G\cos(2n\varphi) + b - a)\right].
\end{aligned}$$

In the above formulae $F_1 = \operatorname{Re}(I_1)/2\sqrt{2G}$ and $F_2 = \operatorname{Im}(I_1)$. Proceeding in the way similar to the previous case we can also construct polynomial in the momenta first integrals for positive rational n.

Obtained results can be rewritten immediately for Hamiltonian systems with indefinite flat form of kinetic energy, which in polar coordinates are given by the following Hamilton function

$$H = \frac{1}{2}\left(p_r^2 - \frac{p_\varphi^2}{r^2}\right) + V, \tag{4.21}$$

with potential

$$V = \frac{a}{r^2\cosh^2(n\varphi)} + \frac{b}{r^2\sinh^2(n\varphi)}. \tag{4.22}$$

Coordinates (r, φ) are related to the Cartesian coordinates by the formulae

$$q_1 = r\cosh(\varphi), \qquad q_2 = r\sinh(\varphi). \tag{4.23}$$

This system is separable in (r, φ) coordinates with first integral

$$G = \frac{1}{2}p_\varphi^2 - \frac{a}{\cosh^2(n\varphi)} - \frac{b}{\sinh^2(n\varphi)}. \tag{4.24}$$

One more additional first integral has the form

$$I_1 = r^{-2n}(\sqrt{2G} - rp_r)^{2n}\left[\sqrt{2G}\sinh(2n\varphi)p_\varphi + 2(G\cosh(2n\varphi) + a + b)\right]. \tag{4.25}$$

It can be obtained either by a direct integration, or from integral (4.19) by substitutions

$$\varphi \to \mathrm{i}\varphi, \qquad p_\varphi \to -\mathrm{i}p_\varphi \quad G \to -G, \quad \sqrt{G} \to \mathrm{i}\sqrt{G}, \quad b \to -b. \tag{4.26}$$

One can construct also another first integral

$$I_2 = r^{-2n}(\sqrt{2G} + rp_r)^{2n}\left[\sqrt{2G}\sinh(2n\varphi)p_\varphi - 2(G\cosh(2n\varphi) + a + b)\right] \tag{4.27}$$

from integral (4.19) choosing the other square root of $-G$, i.e. making the substitution $\sqrt{G} \to -\mathrm{i}\sqrt{G}$ in (4.19). Then, for $n \in \mathbb{N}$, either $I_1 + I_2$, or $I_1 - I_2$, is polynomial in momenta first integral. In general case for positive rational $n = n_1/n_2$, either $F_1 := I_1^{n_2} + I_2^{n_2}$, or $F_1 := I_1^{n_2} + I_2^{n_2}$ is a first integral which is polynomial in momenta.

4.3 Other form of the additional integral, polynomial in momenta

In the previous section we showed that the additional first integral is polynomial in polar momenta p_φ and p_r. Here we show an approach which allows to demonstrate that this integral is expressible in terms of polynomials closely related with the Chebyshev polynomials. The obtained form of the first integral shows that is rational in Cartesian variables (q_1, q_2, p_1, p_2) and polynomial in momenta (p_1, p_2).

Let us introduce double spherical coordinates

$$q_1 = r\cos\varphi, \quad q_2 = r\sin\varphi, \qquad p_1 = p\cos\psi, \quad p_2 = p\sin\psi. \tag{4.28}$$

Let us consider for example natural Hamiltonian with potential (1.10) for $a = 1$ and $b = 0$. In polar coordinates (4.28) Hamiltonian takes the form

$$H = \frac{1}{2}p^2 + \frac{1}{r^2\cos^2(n\varphi)},$$

and Hamiltonian equations transform into

$$\begin{aligned}
\dot{r} &= p\cos(\varphi - \psi), \\
\dot{\varphi} &= -\frac{p}{r}\sin(\varphi - \psi), \\
\dot{p} &= -\frac{2}{r^3\cos^3(n\varphi)}\left[\frac{n-1}{2}\cos\left((n+1)\varphi - \psi\right) - \frac{n+1}{2}\cos\left((n-1)\varphi + \psi\right)\right], \\
\dot{\psi} &= -\frac{2}{pr^3\cos^3(n\varphi)}\left[\frac{n-1}{2}\sin\left((n+1)\varphi - \psi\right) + \frac{n+1}{2}\sin\left((n-1)\varphi + \psi\right)\right].
\end{aligned} \tag{4.29}$$

Let us note that transformation (4.28) is not canonical. In these coordinates Jacobi first integral takes the form

$$I_0 = r^2p^2\sin^2(\varphi - \psi) + \frac{2}{\cos^2(n\varphi)}.$$

Let us look for an additional first integral of the form

$$I = p^n \sin(n\psi) + \sum_{i=1}^{[n/2]} (-2)^i \frac{p^{n-2i}}{r^{2i}\cos^{2i}(n\varphi)} \sum_{m=i-1}^{n-1-i} a_{i,m} \sin[2(n-m-1)\varphi - (n-2m-2)\psi],$$

where a_{im} are unknown constant coefficients. Substitution of these formulae into condition $\dot{I} = 0$ yields the following recurrence equation on a_{im}

$$\begin{aligned}&[i(n-1)+m+1]a_{i+1,m+1} - [i(n+1)+m+2]a_{i+1,m} = (n+1)(n-m-i-1)a_{i,m}\\ &-(n-1)(m-i+2)a_{i,m+1}, \qquad i = 1, \ldots, \left[\frac{n-2}{2}\right], \quad m = i, \ldots, n-i-2.\end{aligned} \tag{4.30}$$

It has the following solution

$$a_{i,m} = \frac{(m+2-i)_i(n-m-i)_{i-1}}{(1)_{i-1}(2)_{i-1}}, \tag{4.31}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1)$$

is the Pochhammer symbol.

We define homogeneous polynomials f_n and g_n and F_n, G_n in the following way

$$(q_1 + \mathrm{i}q_2)^n = f_n + \mathrm{i}g_n, \qquad (p_1 + \mathrm{i}p_2)^n = F_n + \mathrm{i}G_n.$$

From this definition the connection of polynomials f_n, g_n as well as F_n, G_n with Chebyshev polynomials is obvious, see e.g. Section 2 in Freudenburg and Freudenburg [2009].

Then the first integral I in the Cartesian coordinates has the form

$$I = G_n + \sum_{i=1}^{[n/2]} \frac{(-2)^i}{f_n^{2i}} \sum_{m=i-1}^{n-1-i} a_{i,m} r^{2[(i-1)(n-1)+m]} p^{2(m-i+2)} S_{i,m},$$

where

$$S_{i,m} := \operatorname{Im}\left[(p_1 - \mathrm{i}p_2)^{n-2m-2}(q_1 + \mathrm{i}q_2)^{2(n-m-1)}\right].$$

Let us notice that

$$\operatorname{Im}\left[(p_1 - \mathrm{i}p_2)^{\alpha}(q_1 + \mathrm{i}q_2)^{\beta}\right] = F_\alpha g_\beta - G_\alpha f_\beta.$$

Hence the above first integral is rational in (q_1, q_2, p_1, p_2) and polynomial in (p_1, p_2).

In the similar way one can treat the general potential (1.10), however in this case the form of the first integral is much more complicated.

4.4 Generalisation of TTW system to a system on $\mathbb{S}^2$ and $\mathbb{H}^2$

In this section we consider systems given by the Hamiltonian function (1.18) with potential

$$V_n^{(\kappa)}(r,\varphi) := \frac{1}{\mathrm{S}_\kappa^2(r)} U(\varphi), \tag{4.32}$$

where

$$U(\varphi) = \frac{a}{\cos^2(n\varphi)} + \frac{b}{\sin^2(n\varphi)}.$$

This is a natural generalisation of the system considered in the previous subsection onto the spaces with a constant non-zero curvature[2]. It is not difficult to show that for this potential Lemma 4.1 applies. That is, if the potential is super-integrable, then n is non-zero rational number. We show that in fact those potentials are super-integrable, i.e., that the necessary conditions of Theorem 1.1 are also sufficient. To this end it is enough to perform the explicit integration similar to that done in the previous subsections. It gives the following form of the first integral

$$I = \left(\sqrt{2G}\,\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} + \mathrm{i}\,p_r\right)^{2n} \left(\sqrt{G}p_\varphi \sin(2n\varphi) + \mathrm{i}\sqrt{2}(G\cos(2n\varphi) + b - a)\right), \tag{4.33}$$

and the first integral G takes the form

$$G = \frac{1}{2}p_\varphi^2 + U(\varphi).$$

Assuming that $n \in \mathbb{N}^\star := \mathbb{N} \setminus \{0\}$, $a, b \in \mathbb{R}$, and taking real and imaginary parts of (4.33) we obtain the following explicit forms of first integrals

$$\begin{aligned}
I_1 = \sqrt{G}\,\mathrm{Re}\, I = {} & \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} (2G)^{n-j} \left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)^{2n-2j-1} p_r^{2j} \left[G p_\varphi \sin(2n\varphi) \frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} \right. \\
& \left. - \frac{2(n-j)}{2j+1} p_r (G\cos(2n\varphi) + b - a) \right], \\
I_2 = \frac{\mathrm{Im}\, I}{\sqrt{2}} = {} & \sum_{j=0}^{n} (-1)^j \binom{2n}{2j} (2G)^{n-j} \left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)^{2n-2j-1} p_r^{2j} \left[\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} (G\cos(2n\varphi) + b - a) \right. \\
& \left. + \frac{n-j}{2j+1} p_r p_\varphi \sin(2n\varphi) \right].
\end{aligned}$$

Analogous calculations can be repeated for potential

$$U(\varphi) = \frac{a}{\cosh^2(n\varphi)} + \frac{b}{\sinh^2(n\varphi)}.$$

The Jacobi first integral for it takes the form (4.24) and this one obtained from separation of variables in polar coordinates is

$$\begin{aligned}
I &= \left(\sqrt{2G}\,\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)} + p_r\right)^{2n} \left(-\sqrt{G}p_\varphi \sinh(2n\varphi) + \sqrt{2}(G\cosh(2n\varphi) + a + b)\right) \\
&= \left(-\sqrt{G}p_\varphi \sinh(2n\varphi) + \sqrt{2}(G\cosh(2n\varphi) + a + b)\right) \sum_{j=0}^{2n} \binom{2n}{j} (2G)^{(2n-j)/2} \left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)^{2n-j} p_r^j.
\end{aligned}$$

[2]During the final stage of preparation the work of Kalnins et al. [2010c] appeared where the reader will find other examples of super-integrable systems on constant curvature spaces.

It can be also obtained directly from (4.33) using the substitutions (4.26). One can also construct first integral using different root of $-G$ i.e. making the substitution $\sqrt{G} \to -\mathrm{i}\sqrt{G}$ in (4.33) and we obtain the following form

$$I = \left(\sqrt{G}p_\varphi \sinh(2n\varphi) + \sqrt{2}(G\cosh(2n\varphi) + a + b)\right)\sum_{j=0}^{2n}(-1)^j\binom{2n}{j}(2G)^{(2n-j)/2}\left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)^{2n-j}p_r^j.$$

Proceeding in the way similar to the previous cases we can also construct polynomial in the momenta first integrals for positive rational n.

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